

# Critical statistics in quantum chaos and Calogero-Sutherland model at finite temperature

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## Abstract

We investigate the spectral properties of a generalized GOE (Gaussian Orthogonal Ensemble) capable of describing critical statistics. The joint distribution of eigenvalues of this model is expressed as the diagonal element of the density matrix of a gas of particles governed by the Calogero-Sutherland Hamiltonian (C-S). Taking advantage of the correspondence between C-S particles and eigenvalues, we show that the number variance of our random matrix model is asymptotically linear with a slope depending on the parameters of the model. Such linear behavior is a signature of critical statistics. This random matrix model may be relevant for the description of spectral correlations of complex quantum systems with a self-similar/fractal Poincaré section of its classical counterpart. This is shown in detail for two examples: the anisotropic Kepler problem and a kicked particle in a well potential. In both cases the number variance and the  $\Delta_3$ -statistic is accurately described by our analytical results.

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# 1 Introduction

Random matrix ensembles (RME) are an invaluable tool in describing the level statistics of complex quantum systems. Typically, their range of applicability for disordered systems is determined by the Thouless energy which, in the metallic phase, is much larger than the average level spacing. In the neighborhood of a localization-delocalization transition the Thouless energy is of the order of the average level spacing and the wavefunctions become multifractal. The usual random matrix ensembles are no longer applicable. Recently, new random matrix ensembles [1, 2, 3, 4, 5, 6, 7, 8] depending on additional parameters have been proposed to describe spectral correlations in this critical case. These new models for critical statistics have been successfully utilized to describe the spectral correlations of a disordered system at the Anderson transition in three dimensions [9, 10], two dimensional Dirac fermions in a random potential [11], the quantum Hall transition [12] and of the QCD Dirac operator in a liquid of instantons [13, 7].

There are two different types of models for critical statistics. In the first one, deviations from Wigner-Dyson statistics are obtained by adding a symmetry breaking term to the GUE [2, 7]. The model is solved by mapping it to a non-interacting Fermi gas of eigenvalues. The second one [4] makes use of soft confining potentials and is solved exactly by means of  $q$ -orthogonal polynomials. Both models lead to the same spectral kernel for small deviations from the GUE. Based on this observation it was conjectured [1] that critical statistics is universal. However, the origin of the critical kernel is different in both cases. In models based on a soft confining potential the critical kernel is obtained from a nontrivial unfolding. In models with an explicit symmetry breaking term, deviations from Wigner-Dyson statistics arise because the long range correlations between the eigenvalues are exponentially suppressed [14]. We remark that the extent of universality in critical statistics is still under debate. For instance, for chiral ensembles, the correlation functions of a model based on a symmetry breaking term [7] and a model with a soft confining potential [15] are different. Only the former one reproduces critical statistics.

Critical random matrix models for orthogonal and symplectic ensembles have recently [16] been reported in the literature. Tselik and Kravtsov have obtained asymptotic expressions for the critical two level spectral function from a generalized ensemble of random banded matrices. An exact expression for the two level critical spectral function was conjectured in [16] for orthogonal and symplectic ensembles. In the context of the Anderson model, a similar result was conjectured by Nishigaki [10].

In order to describe the spectral correlations of certain pseudo-integrable billiards with dynamics intermediate between chaotic and integrable, Bogomolny and coworkers [17] have introduced a short range plasma model that interpolates between Poisson statistics and Wigner-Dyson statistics. The joint distribution of eigenvalues in [17] is given by the

classical Dyson gas with the logarithmic pairwise interaction restricted to a finite number  $k$  of nearest neighbors. Analytical solutions are available for general  $k$  and symmetry class. It turns out that this short-range plasma model reproduces the typical characteristics of critical statistics like a linear number variance, with a slope depending on  $k$  and, asymptotically, an exponential decay of the nearest neighbor spacing distribution. However the two models are not identical. In critical random matrix models based on a symmetry breaking term, the joint distribution of eigenvalue can be considered as an ensemble of free particles at finite temperature with a nontrivial statistical interaction. The statistical interaction resembles the Vandermonde determinant, and the effect of finite temperature is to suppress the correlations of distant eigenvalues. In [17] this suppression is abrupt, in contrast to critical statistics, where the effect of the temperature is smooth. For further details we refer to [14].

In this paper we introduce a generalized GOE based on the addition of a symmetry breaking term to an invariant Gaussian probability distribution along the lines of [2] for the GUE. We show that the joint eigenvalue distribution of this model coincides with the diagonal element of the density matrix of a gas of particles governed by the Calogero-Sutherland (C-S) Hamiltonian. Using this identification we calculate the asymptotic behavior of the number variance from the susceptibility of the C-S partition function. Because the Itzykson-Zuber integral for  $\beta = 1$  is unknown, a direct calculation of the correlation functions is not possible. To obtain analytical results we invoke the Kravtsov-Tsvelik conjecture, which states that the finite temperature modifications of the correlation functions arise only through the known finite temperature modifications of the kernel for  $\beta = 2$ . The validity of this conjecture is tested in two different ways. First, we show that it is in agreement with a conformal calculation of the asymptotic behavior of the two point correlation function. Second, the asymptotic behavior of the number variance according to the Kravtsov-Tsvelik conjecture agrees with the behavior of the susceptibility in the grand canonical ensemble. One of the main aims of this article is to show that critical statistics describes the spectral correlations of time-reversal invariant quantum systems with a corresponding classical phase space that has a global self-similar, fractal structure. This is shown in two examples, a kicked particle in a potential well and the anisotropic Kepler problem, by comparing the two-point level correlations with the analytical formula of our critical Random Matrix Model.

The critical Random Matrix Model and its relations with the C-S model is discussed in section 2. In section 3, we review the Kravtsov-Tsvelik conjecture for the density-density correlation function of the C-S model in the low temperature limit. The validity of this conjecture is discussed in section 4. In section 5 we show that the level correlations of a kicked particle in a potential well and of the anisotropic Kepler problem are described by the Kravtsov-Tsvelik conjecture. Concluding remarks are made in section 6.

## 2 Definition of the Model

A Random Matrix Model for Hermitian matrices with critical eigenvalue statistics was introduced in [2]. Although it is straightforward to generalize this model to the class of the Gaussian Orthogonal Ensemble, the absence of an explicit result for the integral over orthogonal matrices makes its analysis far more complicated. The model we study is defined by the joint probability distribution

$$P(S, b) = \int dM e^{-\frac{1}{2}\text{Tr}SS^T} e^{-\frac{b}{2}\text{Tr}[M,S][M,S]^T}. \quad (1)$$

Here, the  $N \times N$  matrices  $S$  and  $M$  are real symmetric and orthogonal, respectively, and the integration measure  $dM$  is the Haar measure. From the invariance of  $dM$  it follows that  $P(S, b)$  is a function of the eigenvalues of  $S$  only. If the eigenvalues of  $S$  are denoted by  $x_k$  the joint eigenvalue distribution is given by

$$\rho(x_1, \dots, x_N) = \Delta(\{x_k\}) \int dM e^{-\frac{1}{2}(2b+1)\sum_k x_k^2 + b\sum_{k,l} M_{kl}^2 x_k x_l}. \quad (2)$$

where the Vandermonde determinant is defined by

$$\Delta(\{x_k\}) = \prod_{k < l} (x_k - x_l). \quad (3)$$

Let us now consider the harmonic oscillator Hamiltonian

$$\hat{H} = -\nabla_S^2 + \frac{1}{4}\omega^2 \text{Tr}S^2 \quad (4)$$

where the Laplacian for symmetric matrices  $S$  is given by

$$\nabla_S^2 = \sum_{i=1}^N \frac{\partial^2}{\partial S_{ii}^2} + \frac{1}{2} \sum_{i < j}^N \frac{\partial^2}{\partial S_{ij}^2}. \quad (5)$$

This Hamiltonian is the sum of  $N(N+1)/2$  independent harmonic oscillators with imaginary time propagator given by [18]

$$\langle S | e^{-\tau \hat{H}} | S' \rangle = \left( \frac{\omega}{4\pi \sinh \omega \tau} \right)^{N(N+1)/4} e^{-\frac{\omega}{4 \sinh \omega \tau} [(\text{Tr}S^2 + \text{Tr}S'^2) \cosh \omega \tau - 2\text{Tr}SS']}. \quad (6)$$

Since the Laplacian,  $\nabla_S^2$ , is the sum of a radial piece, depending only on the eigenvalues of  $S$ , and an angular piece, depending only on the orthogonal matrix  $M_S$  that diagonalizes  $S$ ,

$$\nabla_S^2 = \frac{1}{\Delta(\{x_k\})} \sum_{i=1}^N \frac{\partial}{\partial x_i} \Delta(\{x_k\}) \frac{\partial}{\partial x_i} + \nabla_{M_S}^2, \quad (7)$$

the matrix element  $\langle S|e^{-\tau\hat{H}}|S'\rangle$  factorizes into a radial piece and an angular piece. After integration over the angular degrees of freedom and putting the eigenvalues of  $S'$  equal to the eigenvalues of  $S$  we obtain

$$\langle x_1, \dots, x_N | \Delta^{1/2}(\{x_k\}) e^{-\tau H_{\text{rad}}} \Delta^{-1/2}(\{x_k\}) | x_1, \dots, x_N \rangle = C \int dM e^{-\frac{\omega}{2 \sinh \omega \tau} [\text{Tr} S^2 \cosh \omega \tau - \text{Tr} S M S M^T]}, \quad (8)$$

where the integral over the angular matrix element has been absorbed in the normalization constant  $C$ . If we make the identification

$$\frac{\omega}{\sinh \omega \tau} = 2b \quad \text{and} \quad \frac{\omega \cosh \omega \tau}{\sinh \omega \tau} = 2b + 1, \quad (9)$$

the r.h.s. of this equation is exactly the joint probability distribution (2). We thus have shown that the joint probability distribution of our model is given by the diagonal matrix element of the density matrix of the Hamiltonian

$$\hat{H} = \Delta^{1/2}(\{x_k\}) H_{\text{rad}} \Delta^{-1/2}(\{x_k\}). \quad (10)$$

Using the identity,

$$\Delta^{-1/2}(\{x_k\}) \sum_{i=1}^N \frac{\partial}{\partial x_i} \Delta(\{x_k\}) \frac{\partial}{\partial x_i} \Delta^{-1/2}(\{x_k\}) = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{1}{4} \sum_{k \neq l} \frac{1}{(x_k - x_l)^2}, \quad (11)$$

we find the Hamiltonian

$$\hat{H} = - \sum_j \frac{\partial^2}{\partial x_j^2} - \frac{1}{4} \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} + \frac{\omega^2}{4} \sum_j x_j^2. \quad (12)$$

This Hamiltonian corresponds to the Calogero-Sutherland model [19, 20]

$$\hat{H}_{C-S} = - \sum_j \frac{\partial^2}{\partial x_j^2} + \frac{\lambda}{2} \left( \frac{\lambda}{2} - 1 \right) \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} + \frac{\omega^2}{4} \sum_j x_j^2. \quad (13)$$

with  $\lambda = 1$  and fermionic boundary conditions. We have thus shown that the joint eigenvalue distribution of the model (1) is given by the diagonal matrix elements of the  $N$ -particle density matrix of the Calogero-Sutherland model at an inverse temperature  $\tau$  given by (9).

The normalized eigenfunctions of the Calogero-Sutherland Hamiltonian (12) can be labeled in terms of the partitions of integers denoted by  $\kappa$  (see next section). If  $\lambda_\kappa$  and  $\Psi_\kappa(x_1, \dots, x_N, \omega)$  are the eigenvalues and eigenfunctions of the C-S Hamiltonian, respectively, the joint eigenvalue probability distribution is given by

$$\rho(x_1, \dots, x_N) = C' \sum_{\kappa} e^{-\lambda_\kappa \tau} \Psi_\kappa(x, \omega) \Psi_\kappa(x, \omega) \quad (14)$$

where  $C'$  is a constant and  $x = x_1, \dots, x_n$ . The  $\Psi_\kappa(x, \omega)$  can be expressed in terms of the generalized Hermite polynomials [30]

$$\Psi_k(x, \omega) = \frac{1}{\sqrt{N_\kappa}} e^{-\frac{1}{4}\omega \sum_k x_k^2} \Delta^{1/2}(\{x_l\}) H_k(x\sqrt{\omega/2}, 2) \quad (15)$$

where  $N_\kappa$  is a normalization constant and the eigenvalue is given by

$$\lambda_\kappa = \omega|\kappa|. \quad (16)$$

We point out that the above relation between the Euclidean propagator in symmetric spaces and the C-S model at finite temperature can be extended to all nine other symmetry classes in the Cartan classification of large families of symmetric spaces [21]. In essence, the radial part of the Laplacian in the symmetric space corresponds to a C-S type Hamiltonian. For a classification of C-S Hamiltonians based on the symmetry class we refer to [22].

Finally, let us mention that the interpolating role (between RMT and the Poisson ensemble) of  $b$  can be inferred directly from (1). Using the invariance of the measure, the integral over  $M$  can be replaced by an integral over the eigenvalues of  $M$ . For  $b \rightarrow \infty$ , this partition function is dominated by matrices  $S$  that commute with arbitrary diagonal orthogonal matrices. This set of matrices is the ensemble of diagonal symmetric matrices also known as the Poisson ensemble with uncorrelated eigenvalues. Critical statistics is obtained in the thermodynamic limit if the parameter  $b$  is scaled as

$$b = h^2 N^2. \quad (17)$$

Wigner-Dyson statistics is found for a weaker  $N$ -dependence of  $b$  whereas a stronger  $N$ -dependence leads to Poisson statistics. This transition can also be understood in terms of the C-S model at finite temperature. At zero temperature, the probability density of the ground state of the C-S model (12) coincides with the joint probability distribution of the Gaussian Orthogonal Ensemble. In the high temperature limit,  $\tau \rightarrow 0$ , the positions of the particles become uncorrelated and the statistics of the associated matrix model is Poisson.

To recapitulate, we have traded the problem of performing an integral over the orthogonal group by the physical task of finding the diagonal element of the density matrix of an ensemble of particles governed by the C-S Hamiltonian.

## 2.1 Excited states and Zonal Polynomials

In this section we discuss explicit solutions of the excited eigenfunctions of the C-S Hamiltonian (12) and argue to what extent they are useful for the evaluation of correlation functions from the joint eigenvalue distribution (2).

The probability density of the ground state of the C-S model (12) coincides with the joint probability distribution of the Gaussian Orthogonal Ensemble [19]. This observation together with the conjecture of the solvability of the C-S model was already made in the pioneering articles of Calogero [20] and Sutherland [19]. Later, Sutherland [23] obtained a non-orthogonal set of solutions. The problem of finding a set of orthogonal solutions for these excited states was recently solved by Forrester, Ha and Serban [24, 25, 26] who expressed the wavefunctions of the excited states in terms of the symmetric Jack polynomials [27]. For the special values of the coupling constant related to GOE and GSE the Jack polynomials have a geometrical interpretation and are usually called zonal polynomials [28]. Unfortunately, there is no closed formula neither for the Jack nor for the zonal polynomials [28]. Since explicit calculations rely on recurrence relations, numerical work is needed to evaluate polynomials of high degree. In our case, due to the harmonic potential, the excited states are given by the generalized Hermite (or Hidden-Jack) polynomials [29] which can be expressed in terms of Jack polynomials [29, 30].

The generalized Hermite polynomials in (14) can be expressed in terms of zonal polynomials by means of a Mehler type formula [30].

$$\begin{aligned} \rho(x_1, \dots, x_N) &\propto e^{-\frac{1}{2}\omega \sum_k x_k^2} \Delta(\{x_k\}) \sum_{\kappa} \frac{e^{-\lambda_{\kappa}\tau}}{N_{\kappa}} H_{\kappa}(x\sqrt{\omega/2}, 2) H_{\kappa}(x\sqrt{\omega/2}, 2) \\ &\propto \Delta(\{x_k\}) e^{-\frac{1}{2}\omega \coth(\tau\omega) \sum_i x_i^2} \sum_{\kappa} \frac{1}{|\kappa|!} \frac{C_{\kappa}^{(2)}(x\sqrt{2\omega}/\sqrt{1-e^{-2\omega\tau}}) C_{\kappa}^{(2)}(x\sqrt{\omega/2}e^{-\tau\omega}/\sqrt{1-e^{-2\omega\tau}})}{C_{\kappa}^{(2)}(1^N)} \end{aligned} \quad (18)$$

where the  $C_{\kappa}^{(2)}(x)$  are the symmetric Jack polynomials as defined in [30],  $x = x_1, \dots, x_n$  and  $\kappa$  labels the partitions of the integers (there is a polynomial for each partition) and the sum runs over all the partitions. Furthermore,  $\tau$  and  $\omega$  are related to  $b$  through equation (9).

The kernel

$$F_0(x, y) \equiv \sum_{\kappa} \frac{1}{|k|!} \frac{C_{\kappa}^{(2)}(2x) C_{\kappa}^{(2)}(y)}{C_{\kappa}^{(2)}(1^N)} \quad (19)$$

has been studied extensively [28, 30]. Using the result for the  $\tau \rightarrow 0$  limit of the kernel [30],

$$F_0(x/\sqrt{\tau}, y/\sqrt{\tau}) \propto \frac{1}{\sqrt{\Delta(\{x_k\})\Delta(\{y_k\})}} \prod_{j=1}^N e^{\tau x_j y_j / 2} \quad (20)$$

one easily shows that the eigenvalues of our model are uncorrelated in the high temperature limit.

In the zero temperature limit (GOE), the joint distribution can be represented as a quaternionic determinant and the integrations can be performed by means of a "kernel

relation". At nonzero temperatures, the kernel  $F_0(x, y)$  satisfies the "kernel relation" [30],

$$\int d\mu(y) F_0(2y, z) F_0(2y, x) \propto F_0(2z, x) e^{-\frac{1}{2}\omega \sum_i (z_i^2 + x_i^2)}, \quad (21)$$

where  $d\mu(y) = \prod_{i=1}^N e^{-\frac{\omega}{2}y_i^2} \Delta(\{y_k\}) dy_1 \dots dy_N$ , but it is not known<sup>1</sup> whether the joint eigenvalue distribution can be expressed in terms of quaternionic determinants. Further progress could rely on exploiting the orthogonality relations [28] verified by the Zonal polynomials. If proceeding so, the partition function associated with (1) can be evaluated exactly. As expected, it coincides with the one encountered for particles obeying fractional statistics in one dimension [31]. Because of technical problems, the same strategy fails for the spectral density and higher order spectral correlation functions.

For the special case of  $N = 2$ , an explicit calculation of the joint distribution of eigenvalues (2) coincides with the result obtained by using Zonal polynomials techniques [32].

$$\rho(x_1, x_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(x_1^2 + x_2^2)} I_0\left(\frac{b}{2}(x_1 - x_2)^2\right) e^{-\frac{b}{2}(x_1 - x_2)^2} |x_1 - x_2| \quad (22)$$

where  $I_0$  is the Bessel function of complex argument.

In conclusion, we have expressed the joint distribution for the eigenvalues of our model in terms of Jack polynomials but we have not succeeded to derive explicit expressions for the correlation functions.

### 3 Critical spectral kernel and density-density correlations of the C-S model at finite temperature

We recall that the two level spectral function of our model is identical to the density-density correlation function of the C-S model (13) for  $\lambda = 1$ . In [16] it was conjectured that the low temperature limit of the connected density-density correlation function of the C-S model at  $\lambda = 1$  is given by,

$$\langle \rho(x) \rho(0) \rangle_T - \langle \rho(x) \rangle_T \langle \rho(0) \rangle_T = R_{2,c}^T(x, 0) = -\bar{K}_T^2(x, 0) - \left( \frac{d}{dx} \bar{K}_T(x, 0) \right) \int_x^\infty \bar{K}_T(t, 0) dt \quad (23)$$

where

$$\bar{K}_T(x, 0) = T \frac{\sin(\pi x)}{\sinh(\pi x T)} \quad (24)$$

is the kernel of the C-S model (13) for  $\lambda = 2$ . The temperature  $T = \frac{\pi h}{2}$  and  $h$  is related to  $\tau$  and  $\omega$  through (17) and (9). This result is valid in a normalization such that the average density of the particles is equal to unity.

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<sup>1</sup>Some interesting results for small values of  $N$  were obtained in [33].



The idea is that, based on the Luttinger liquid nature of the C-S model [36], the known relation between the density-density correlation of the C-S for  $\lambda = 1$  at zero temperature and the spectral correlations of the GOE can be extended to finite low temperature. One simply replaces the kernel at zero temperature, which physically corresponds to free fermions for all invariant RMT ensembles<sup>2</sup>, by its finite temperature analogue [2] given by,

$$K^T(x, y) = \sum_n \frac{\psi_n(x)\psi_n^\dagger(y)}{1 + z^{-1}e^{\tau E_n}} \quad (25)$$

where  $\psi_n$  are the single particle wave functions for free fermions and  $E_n = \omega n$ . The fugacity  $z$  for the free fermion distribution in (25) is determined by the total number of particles through,

$$N = \int dx \rho(x) = \sum_{n=0}^{\infty} \frac{1}{1 + z^{-1}e^{\omega\tau n}}. \quad (26)$$

where  $\rho(x) = K^T(x, x)$  is the average spectral density. In the low temperature limit,  $\omega\tau \ll 1$ , we find

$$z^{-1} = \frac{1}{e^{\tau\omega N} - 1} \quad (27)$$

In eq. (9) the quantities  $\tau$  and  $\omega$  have been related to the parameter  $h$  of the matrix model (1). In the large  $N$  limit these relations simplify to

$$\begin{aligned} \omega\tau &\sim \frac{1}{hN} \\ \omega &\sim 2hN. \end{aligned} \quad (28)$$

and for  $h \ll 1$  the fugacity is given by  $z \sim e^{1/h}$ . To obtain the average particle density for  $x \rightarrow 0$  we can approximate the single particle wave function by plane waves with energy given by  $k^2$ .

$$\rho(0) = \frac{1}{\pi} \int_0^\infty dk \frac{1}{1 + z^{-1}e^{\tau k^2}} \sim \sqrt{2} \frac{N\sqrt{h}}{\pi} \quad (29)$$

The unfolded spectral kernel is thus given by,

$$\bar{K}^T(x, 0) = \frac{K^T(x/\rho(0), 0)}{\rho(0)} = \sqrt{h} \int_0^\infty \frac{\cos(\pi x \sqrt{ht})}{2\sqrt{t}} \frac{1}{1 + z^{-1}e^t} dt. \quad (30)$$

In the low temperature limit  $h \ll 1$ , the above spectral kernel coincides with (24) for  $T = \frac{\pi h}{2}$ .

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<sup>2</sup>For  $\lambda \neq 2$  the particles obey exclusion statistics

The number variance of the eigenvalues near the center of the band is obtained by integrating the two point connected correlation function  $R_{2,c}^T(s, 0)$  including the self-correlations (23),

$$\Sigma^2(L) = L + 2 \int_0^{L/\rho(0)} ds (L-s) R_{2,c}^{T=\pi h/2}(s, 0). \quad (31)$$

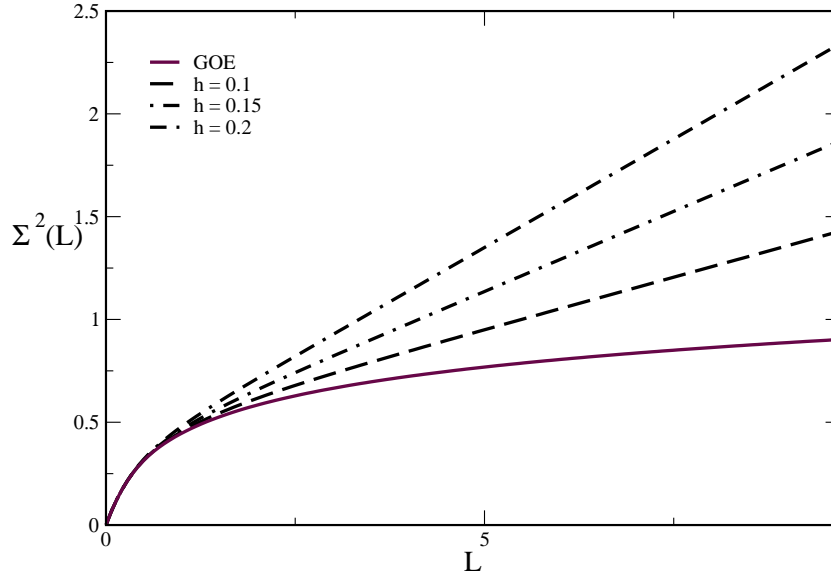


Figure 1: The number variance  $\Sigma^2(L)$  (31) versus  $L$  for  $h = 0.1$ ,  $h = 0.15$  and  $h = 0.2$ . The linear behavior of the number variance is a signature of critical statistics. The slope for  $h \ll 1$  is  $\chi = h$ .

The number variance  $\Sigma^2(L)$  measures the stiffness of the spectrum. The fluctuations are small for the GOE with  $\Sigma^2(L)$  proportional to  $\log(L)$  for  $L \gg 1$ . For the Poisson ensemble, which is an ensemble of diagonal random matrices, the eigenvalues are uncorrelated and  $\Sigma^2(L) = L$ . For critical statistics the number variance is asymptotically proportional to  $\chi L$ . For  $\chi \ll 1$  the slope has been connected with the multifractal dimension  $D_2$  observed in the wave functions of a disordered system undergoing a localization-delocalization transition [34, 35],

$$\chi = \frac{d - D_2}{2d}, \quad (32)$$

where  $d$  is the spatial dimension of the system to be studied. As observed in Figure 1, the number variance of our model is linear for  $L \gg 1$ , with a slope  $\chi = h$  for  $h \ll 1$ .

This linear behavior together with the absence of subleading logarithmic terms in the asymptotic behavior of the number variance suggests that our matrix model describes critical statistics.

## 4 Testing the Kravtsov-Tsvelik Conjecture

Below we discuss two independent methods to test the Kravtsov-Tsvelik conjecture.

### 4.1 Conformal Calculation

We review first [36, 16] how conformal techniques can be utilized to calculate the low temperature large distance asymptotic behavior of the two-point correlation function of the C-S model.

Conformal field techniques [36, 37, 38, 39] can be used to compute the asymptotic behavior of the correlation functions of  $1 + 1$  dimensional systems with a linear gapless spectrum in the limit of large  $N$  number of particles, constant density  $n = N/L$  and low temperature.

In order to identify the conformal theory associated to the low energy excitations of the  $1+1$  dimensional system one needs the value of the conformal anomaly  $c$  of the associated conformal field theory. Usually,  $c$  is obtained from the leading low temperature behavior of the free energy of the system. Then, the conformal weights of the primary fields of the conformal theory must match the leading low energy excitations of the  $1 + 1$  dimensional quantum system. The latter is usually evaluated either numerically or by finite size scaling and Bethe ansatz techniques. This program was carried out for the C-S model by Kawakami and Yang [36]. They found that the low energy collective excitations of the C-S model are described by two quantum numbers:  $\Delta N$  related to excitations that change the number of particles and  $\Delta D$  associated with excitations that move one particle from one Fermi point to the other. The energy,  $E$ , and momentum,  $P$ , of the leading finite size excitations of the C-S model are given by,

$$\begin{aligned} E &= \frac{2\pi}{L} \frac{\lambda}{4} \Delta D^2 + \frac{2\pi}{L} \frac{\lambda}{4} \Delta N^2 + \frac{2\pi(N^+ + N^-)}{L}, \\ P &= 2\pi \Delta D + P_L, \\ P_L &= \frac{2\pi}{L} [\Delta N \Delta D + N^+ + N^-], \end{aligned} \tag{33}$$

where  $P_L$  stands for the momentum of the finite size excitations and  $N^+$  and  $N^-$  label the conformal towers of states (secondary fields) associated to each primary field. Finally, they argued, based on thermodynamics arguments, that the conformal anomaly associated with the C-S model is  $c = 1$ .

In a system with conformal symmetry, the eigenvalues of both the Hamiltonian and the momentum are related to the right/left conformal weights  $x_{n,m}/\bar{x}_{n,m}$  of the primary fields through the following relation,

$$\begin{aligned} x_{n,m} &= \frac{L}{2\pi}[E + P_L], \\ \bar{x}_{n,m} &= \frac{L}{2\pi}[E - P_L], \end{aligned} \quad (34)$$

where we assume that  $E$  and  $P_L$  depend on quantum numbers  $n$  and  $m$ . Therefore, if we manage to find a conformal field theory with  $c = 1$  and eigenvalues of the momentum operator and energy operator given by (33), the correlation functions of the C-S model in the asymptotic limit can be easily evaluated by means of conformal techniques. It turns out that the simplest conformal model with such properties is a free boson compactified on a circle with radius  $R = 1/\sqrt{\lambda}$ .

In general, observables do not have definite conformal dimensions and must be expressed as a linear combination of conformal excitations. Since such conformal fields only describe the excitations close to the ground state one first has to decompose the expansion of observables into “fast” and “slow” modes [41]. The “slow” modes are described by the conformal fields and the fast “ones” correspond to momenta that remain finite in the thermodynamic limit, i.e. to excitations with  $\Delta D \neq 0$ . The density operator can be expanded as

$$\rho(x)_T = \sum_{m,n=-\infty}^{\infty} c_{m,n} e^{i2\pi x n} \psi_{n,0,m}(x)_T \quad (35)$$

where  $\psi_{n,0,m}(x,0)_T$  stands for the primary state ( $\Delta D = n, \Delta N = 0$ ) associated with the above conformal theory, the phase in the expansion represents the momentum of the ground state for  $L \rightarrow \infty$  (fast mode) and the index  $m$  accounts for the contribution of secondary fields. Since the density operator does not change the number of particles only excitations with  $\Delta N = 0$  contribute to the expansion. The coefficients  $c_{m,n}$  are found from the zero temperature limit (*GOE*).

The density-density correlations at finite temperature can be easily obtained using the known result for the correlation functions of the conformal fields  $\psi_{n,0,m}(x,0)_T$ . The first terms of the conformal prediction for the density-density correlations of the C-S model at  $\lambda = 1$  in the low temperature limit are thus given by

$$R_2^T(x,0) = \langle \rho(x)\rho(0) \rangle_T \sim \frac{T^2}{\sinh^2(\pi x T)} - \frac{T^4}{2} \frac{\cos(2\pi x)}{\sinh^4(\pi x T)} - \frac{3}{2} \frac{T^4}{\sinh^4(\pi T x)} \cdots \quad (36)$$

With a temperature as given by the Kravtsov-Tselik conjecture, i.e.  $T = \frac{\pi h}{2}$ , the conformal result coincides with the asymptotic expansion of the conjectured result (23). Since both results have been obtained by using completely different methods, this calculation

supports the validity of the kernel (24) in the low temperature, long distance limit.

## 4.2 Susceptibility

The slope of the large distance asymptotic behavior of the number variance is determined by the isothermal susceptibility of the C-S model which can be obtained from the C-S partition function. On the other hand, this slope is determined by an integral of the unfolded two-point cluster function  $Y_2(r)$  according to

$$\Sigma^2(n) \sim n(1 - \int_{-\infty}^{\infty} Y_2(r) dr). \quad (37)$$

Therefore, agreement between the conformal calculation and the Kratsov-Tselik conjecture for the large distance asymptotic behavior of the two-point correlation function does not necessarily imply that the asymptotic behavior of the number variance is given by the susceptibility.

The susceptibility  $\chi$  in the grand canonical ensemble which measures the fluctuations of the number of particles in a box of length  $L$ ,

$$\chi = \langle N^2 \rangle - \langle N \rangle^2, \quad (38)$$

can be expressed as

$$\chi = z_1 \frac{d}{dz_1} \langle N \rangle, \quad (39)$$

where  $z_1$  stands for the fugacity and  $\langle N \rangle$  is the average number of particles. Remarkably, the C-S gas for arbitrary statistical coupling  $\lambda$  can still be considered a free gas but with exclusion statistics [31]. Indeed, Sutherland [19] has shown, by using the Bethe Ansatz and a method previously developed by Yang and Yang [40] for the Bose gas with a delta interaction, that the occupation number  $n(k)$  of a gas of C-S particles satisfies the following transcendental equation,

$$(1 - \lambda n(k)/2)^{\lambda/2} (1 + (1 - \lambda/2)n(k))^{1-\lambda/2} = n(k) e^{\tau e(k)} / z_1. \quad (40)$$

For the special case  $\lambda = 1$ , corresponding to the *GOE*, we obtain

$$n(k) = \frac{2}{\sqrt{1 + \frac{4}{z_1^2} e^{2\tau e(k)}}} \quad (41)$$

where  $z_1$  is the fugacity for this distribution function and  $e(k) = k^2$  is the energy of a single particle. To find the relation between the fugacity and the parameter  $h$  in the

Moshe-Neuberger-Shapiro model for  $\lambda = 1$  we have to use the single particle energies corresponding to the C-S model (12). We thus have

$$N = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{1 + \frac{4}{z_1^2} e^{2\tau(k^2 + \frac{1}{4}\omega^2 x^2)}}}. \quad (42)$$

The asymptotic behavior for large  $z_1$  can be obtained easily by changing to polar coordinates. This results in

$$N = \frac{\log z_1^2}{\omega\tau}. \quad (43)$$

Using (9) we then find in the limit of small  $h$ ,

$$z_1 = e^{1/2h}. \quad (44)$$

Since the density of particles is  $x$ -dependent in a harmonic box we calculate the susceptibility for particles in a rectangular box with fugacity given by (44). In this way the susceptibility can be compared to the slope of the number variance which is calculated in the center of the spectrum.

The average number of particles in a box of length  $L$  is given by

$$\langle N \rangle = \frac{L}{\pi} \int_{-\infty}^{\infty} \frac{dk}{(1 + \frac{4}{z_1^2} e^{2\tau k^2})^{1/2}}. \quad (45)$$

If the fugacity is parameterized as  $z_1^2 \equiv e^{2\tau \bar{k}^2}$  we have in the low-temperature limit,

$$\langle N \rangle = 2\bar{k} \frac{L}{\pi}. \quad (46)$$

Then

$$\chi \equiv \langle N^2 \rangle - \langle N \rangle^2 = z_1 \frac{d}{dz_1} \langle N \rangle = \frac{L}{\pi} \int_0^{\infty} dk \frac{8z_1^{-2} e^{2\tau k^2}}{(1 + 4z_1^{-2} e^{2\tau k^2})^{\frac{3}{2}}}.$$

After the change of variable  $\delta k = k - \bar{k}$  and expanding around the Fermi surface we find,

$$\begin{aligned} \chi &= \frac{L}{\pi} \int_{-\infty}^{\infty} d\delta k \frac{8e^{4\tau \bar{k} \delta k}}{(1 + 4e^{4\tau \bar{k} \delta k})^{\frac{3}{2}}} \\ &= \frac{\langle N \rangle}{2\tau \bar{k}^2} = \frac{\langle N \rangle}{2 \log z_1} = h \langle N \rangle. \end{aligned} \quad (47)$$

The above result should be compared with the calculation of the asymptotic behavior of the number variance from the two-point spectral correlation function,

$$\chi \sim \Sigma^2(\langle N \rangle) \quad \text{for} \quad \langle N \rangle \rightarrow \infty, \quad (48)$$

where  $\Sigma^2(\langle N \rangle)$  is defined by

$$\Sigma^2(\langle N \rangle) = \langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle - 2 \int_0^{\langle N \rangle} dr (\langle N \rangle - r) Y_2(r). \quad (49)$$

Here,  $Y_2(r)$  is the unfolded two-point cluster function. According to the Kravtsov-Tsvelik conjecture it is given by

$$Y_2(r) = K^2(r) + \frac{dK(r)}{dr} \int_r^\infty K(r') dr', \quad (50)$$

where  $K(r)$  is the kernel

$$K(r) = \frac{1}{2\pi\bar{\rho}} \int_{-\infty}^\infty dk \frac{\cos(kr/\bar{\rho})}{1 + \frac{1}{z_2} e^{\tau k^2}}, \quad (51)$$

and  $\bar{\rho}$  is the average spectral density. For this cluster function, and, in fact any cluster function that decreases stronger than  $1/r$ , we recover the relation (37) for  $\langle N \rangle \rightarrow \infty$ . With the fugacity parameterized by  $z_2 = e^{\tau \bar{k}^2} = e^{1/h}$  (see section 3) we have in the low-temperature limit

$$K(0) \equiv 1 = \frac{\bar{k}}{\pi\bar{\rho}}. \quad (52)$$

After partial integration of the second term of (50) and using that  $K(0) = 1$  we obtain,

$$\Sigma^2(\langle N \rangle) = \langle N \rangle - 2\langle N \rangle \int_0^\infty (2K^2(r) - K(r)) dr + O(N^0). \quad (53)$$

Integrating by parts and making an expansion about the Fermi surface  $\bar{k}$  results in,

$$K(r) = \frac{\sin \bar{k}r/\bar{\rho}}{4\pi r} \int_{-\infty}^\infty ds \frac{\cos sr/2\tau\bar{k}\bar{\rho}}{\cosh^2 s/2} = \frac{1}{2\bar{k}\bar{\rho}\tau} \frac{\sin(\bar{k}r/\bar{\rho})}{\sinh(\pi r/2\bar{k}\bar{\rho}\tau)}. \quad (54)$$

The integral over  $r$  in (54) can now be performed analytically resulting in the susceptibility

$$\chi = \frac{\langle N \rangle}{\tau k^2} = \frac{\langle N \rangle}{\log z_2} = h \langle N \rangle. \quad (55)$$

This slope is in agreement with the result obtained from the partition function of the C-S model. In agreement with our naive expectation, the value of the slope is a factor 2 larger than the one found for the original Moshe-Neuberger-Shapiro model for  $\lambda = 2$  [2].

In the kernel (51) the momentum integral is weighted by the occupation number which in this case is the Fermi-Dirac distribution. Since the occupation number of the C-S model

for  $\lambda = 1$  is given by (41) it seems more natural to make this choice instead. This results in the kernel

$$K(x) = \frac{1}{\pi \bar{\rho}} \int_{-\infty}^{\infty} dk \frac{\cos(2kx/\bar{\rho})}{\sqrt{1 + \frac{4}{z_1^2} e^{2\tau k^2}}} \quad (56)$$

where  $z_1 = e^{\bar{k}^2 \tau} \equiv e^{1/h'}$  is the fugacity. We choose the normalization of the kernel such that  $K(0) = 1$ . Then the zero temperature limit ( $\tau \rightarrow \infty$ ) of this kernel is the usual sine-kernel,  $\sin \pi x / \pi x$ . Below we show that the conjecture (56) disagrees with both the conformal calculation and the susceptibility (47).

Let us first derive the large  $x$  asymptotic behavior of the kernel (56). In the low temperature limit the average number of particles is again given by  $\langle N \rangle = 2L\bar{k}/\pi\bar{\rho}$ , with normalization condition  $2\bar{k}/\pi\bar{\rho} = 1$ . After partial integration the integral can be rewritten as

$$K(x) = \frac{1}{\pi x} \int_0^{\infty} \frac{8\tau k z_1^{-2} e^{2\tau k^2} \sin(2kx/\bar{\rho})}{(1 + 4z_1^{-2} e^{2\tau k^2})^{3/2}}. \quad (57)$$

In the low temperature limit the integrand is strongly peaked at  $k \approx \bar{k}$ , and the integral can be calculated by a steepest descent approximation

$$\begin{aligned} K(x) &\sim \text{Im} \frac{2}{\pi x} \int_{-\infty}^{\infty} du \frac{4e^{2ix(\bar{k}+u/\tau\bar{k})/\bar{\rho}} e^{4u}}{(1 + 4e^{4u})^{3/2}} \\ &\sim \frac{e^{-3/2}}{3\pi} \text{Im} \left( \frac{\pi i x}{\tau \bar{k}} \right)^{1/2} e^{2i\bar{k}x/\bar{\rho} - \pi x \bar{k}/2\tau - \pi i x (\log 4)/2\tau \bar{k}}. \end{aligned} \quad (58)$$

This asymptotic result is in disagreement with the prediction from the conformal calculation.

Next we compare the asymptotic behavior of the number variance with the susceptibility. The comparison of the susceptibility can again be made by computing the asymptotical behavior of the number variance. For an exponentially decreasing kernel we have previously shown that the asymptotic behavior of the number variance is given by

$$\Sigma_2(\langle N \rangle) = \langle N \rangle - 2\langle N \rangle \int_0^{\infty} (2K^2(x) - K(x)) dx + O(\langle N \rangle^0). \quad (59)$$

where  $K(x)$  is the kernel (56) with average eigenvalue spacing normalized to unity. Using that

$$\begin{aligned} \int_0^{\infty} \frac{\sin \pi a x}{\pi x} dx &= \frac{1}{2}, \\ \int_{-\infty}^{\infty} ds \frac{e^{4s}}{(1 + 4e^{4s})^{3/2}} &= \frac{1}{8}, \end{aligned} \quad (60)$$



one easily shows that in the low temperature limit,

$$\int_0^\infty K(x)dx = \frac{1}{2}. \quad (61)$$

The other integral in (59) can be written as

$$\int_0^\infty K^2(x)dx = \int_0^\infty dx \frac{128}{(\pi x)^2} \int_{-\infty}^\infty dy \int_{-\infty}^y dy' \sin((\bar{k} + \frac{y}{\tau \bar{k}})2x/\bar{\rho}) \sin((\bar{k} + \frac{y'}{\tau \bar{k}})2x/\bar{\rho}) G(y)G(y'). \quad (62)$$

where

$$G(y) = \frac{e^{4y}}{(1 + 4e^{4y})^{3/2}}. \quad (63)$$

The integral over  $x$  can be evaluated using the formula

$$\int_0^\infty dx \frac{\sin(ax) \sin(bx)}{x^2} = \frac{a\pi}{2} \quad \text{for } a < b. \quad (64)$$

This results in

$$\int_0^\infty K^2(r)dr = \frac{1}{2} + 128 \int_{-\infty}^\infty dy \int_{-\infty}^y dy' \frac{y'}{\pi \tau \bar{k} \bar{\rho}} G(y)G(y'). \quad (65)$$

The asymptotic behavior of the number variance is thus given by

$$\begin{aligned} \Sigma^2(\langle N \rangle) &= -512 \frac{\langle N \rangle}{\pi \tau \bar{k} \bar{\rho}} \int_{-\infty}^\infty dy \int_{-\infty}^y dy' y' G(y)G(y') \\ &= -64 \frac{\langle N \rangle}{\pi \tau \bar{k} \bar{\rho}} \int_{-\infty}^\infty dy \frac{y e^{4y}}{(1 + 4e^{4y})^2} \\ &= \frac{2\langle N \rangle}{\pi \tau \bar{k} \bar{\rho}} \log 2 \\ &= \frac{\langle N \rangle}{\log z_1} \log 2. \end{aligned} \quad (66)$$

To obtain the expression after the second equality sign we have performed a partial integration using the identity

$$G(y) = -\frac{1}{8} \frac{d}{dy} \frac{1}{\sqrt{1 + 4e^{4y}}}. \quad (67)$$

This result for the susceptibility differs from the result obtained from the thermodynamic properties of the C-S gas. We conclude that the kernel (56) does not describe the correlations of the critical Random Matrix Model (1).

## 5 Critical statistics and quantum chaos

In this section we introduce the concept of multifractal wave functions in the context of the Anderson transition and show how it may be relevant in the study of deterministic quantum chaotic systems.

By now it has been well established that the appearance of critical statistics at the Anderson transition is intimately related with the multifractal properties of the wave-functions [9, 34, 35, 43]. We wish first to introduce intuitively the concept of multifractal wave functions [42] .

Let us consider the volume of the subset of a box for which the absolute value of the wave function  $\Psi$  is larger than a fixed number  $M$ . If this volume scales as  $L^{d^*}$  (with  $d^* < d$ ), then  $d^*$  is called the fractal dimension  $d^* < d$  of  $\Psi$ . In case the fractal dimension depends on the value of  $M$ , the wave function is said to be multifractal. More formally, multifractality is defined through the inverse participation ratio,

$$I_p = \sum_r \langle |\Psi_n(r)|^{2p} \rangle \propto L^{-D_p(p-1)} \quad (68)$$

where  $\Psi_n$  is the wave function with energy  $E_n$ , and  $D_p < d$  is a set of exponents characterizing the anomalous (multifractal) scaling of the moments of the wave function. We remark that, although confined to fractal subsets of the sample, wave functions of such systems overlap strongly when their energies are close enough [43]. Such strong overlap is responsible for the short-range level repulsion observed at the Anderson transition. It is worthwhile to note that this anomalous scaling has, in principle, a pure quantum mechanical origin. As the density of impurities increases, the deBroglie wavelength of the particles becomes comparable with the mean free path and localization effects start to be relevant. We stress that the classical dynamics of the Anderson transition does not provide us with valuable information to describe quantum spectral correlations. One may wonder to what extent such multifractal behavior may be observed in deterministic quantum chaotic systems.

What has become known as the Bohigas-Giannoni-Schmit conjecture [44] is that generically quantum spectra of classically chaotic systems are correlated according to the Wigner-Dyson random matrix ensembles, whereas spectral correlations of classically integrable systems are close to Poisson statistics. In most cases, by modifying the parameters of the system, a transition from integrable to chaotic dynamics can be observed. If the KAM theorem is applicable, this transition is smooth and both integrable and chaotic regions coexist until the last KAM torus is completely destroyed. Although spectral statistics of such mixed system have been described in terms of banded random matrix models [45], they are believed to be non-generic and different from critical statistics [46].

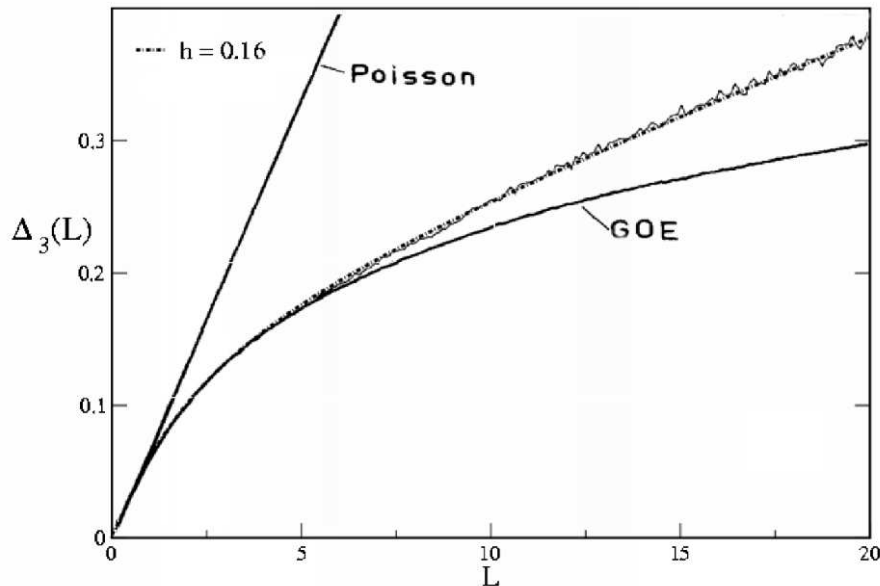


Figure 2: The spectral rigidity  $\Delta_3$  of the anisotropic Kepler problem obtained in [52] versus the prediction of our model for  $h = 0.16$  obtained from (30). Both curves are barely distinguishable. The numerical data are reprinted from Fig 2. in Ref. [52]

The situation is different in cases where the KAM theorem does not apply. In those systems, the invariant KAM curves may not exist at all and small changes in the coupling constant can produce qualitative modifications in the classical phase space. The dynamics is, in general, intermediate between chaotic and integrable. The lack of KAM tori permits a particle explore the full available classical phase space without having full chaotic motion. Such forms of phase space are also known as stochastic webs [47]. In certain cases, the classical phase space becomes increasingly intricate, showing both self-similar and fractal properties [47]. We notice that such a structure is reminiscent of the way that the KAM tori break up into "fractal" orbits of zero dimension [48] (cantori) as the system becomes chaotic. Classically, cantori represent strong obstacles to phase space transport. Our aim is to study the effect, if any, of such self-similar structure in the spectral correlations of the quantum counterpart. Roughly speaking, the influence of cantori on the quantum dynamics will depend on the relation between the size of the cantori and Planck's constant. For cantori smaller than the Planck cell, quantum dynamics cannot resolve the classical fine structure. In this case, cantori act as perfect barriers

to the quantum motion resembling the effect of a classically integrable system and the spectral correlations of the quantum counterpart are close to Poisson statistics. In the intermediate case the situation is less clear. Recently, it has been reported [49] that cantori drive spectral correlations smoothly from Poisson to RMT as the system approaches to the ergodic regime.

Below, we present numerical evidence that deviations from GOE statistics caused by the self similar structure of the classical phase space may be described by critical statistics at least while the deviations from GOE are small.

## 5.1 The Anisotropic Kepler problem

The anisotropic Kepler Hamiltonian

$$H = \frac{1}{2}p_\rho^2 + \frac{1}{2}\gamma p_z^2 - \frac{1}{r} \quad (69)$$

is an interesting example of a non-KAM system undergoing an abrupt chaotic integrable transition. It has been utilized as a model of donor impurities in a semiconductor [50, 51]. Even for small departures from the integrable case,  $\gamma = 1$ , the classical phase space is densely filled with remnants of cantori [52]. Gutzwiller has shown that for  $\gamma < 8/9$  the orbits in phase space can be uniquely represented in terms of symbolic dynamics. Such representation is a signature of hard chaos. Indeed, for  $\gamma < 1/2$  there are no islands of stability in phase space. Furthermore, the measure of the surface of section based on the symbolic dynamics is multifractal with respect to the usual Liouville measure. Since the periodic orbits can be effectively enumerated, the energy levels of the quantum counterpart can be approximately evaluated by means of analytical techniques [53].

A numerical study of the spectral correlations of highly excited states of this system was carried out in [52]. In a basis in which the Hamiltonian has a band structure, they succeeded to obtain up to 5500 energy levels. In Fig. 2 we show their result for the spectral rigidity of the spectrum from level 2501 to 5500. As observed, the deviations from the GOE are very well described by the critical random matrix model (1). Based on the analogy with disordered systems, we conjecture that the wavefunctions of this system are multifractal. We are not aware of numerical results that can confirm or disprove this conjecture.

## 5.2 Kicked particle in a infinite potential well

Recently, in [54], another non-KAM system with similar properties, a kicked particle in a infinite potential well, was studied both quantum mechanically and classically. The

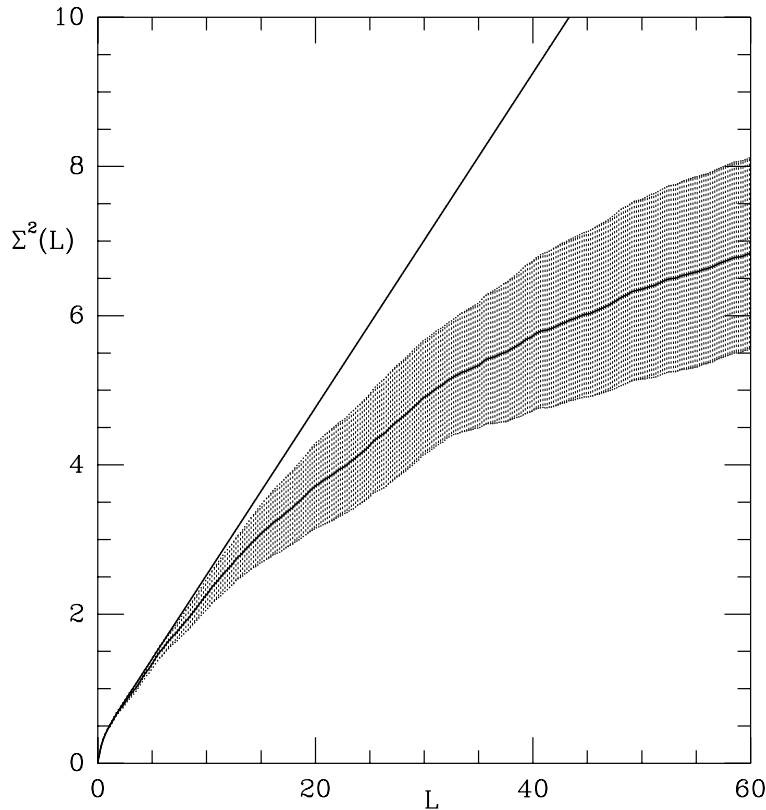


Figure 3: The number variance  $\Sigma^2(L)$  versus  $L$ . The number variance of the quasi-energy levels of a kicked particle in an infinite potential well at  $K = kT = 50$  [54] agrees with the prediction of the critical GOE (31) for  $h = 0.2122$  (upper curve) up to 10 eigenvalues. The downward tendency of the numerical result may be due to finite size effects. The error in the numerical results is indicated by the thickness of the curve. We thank to Baowen Li [54] for kindly providing us with 1024 energy levels to compute the number variance.

Hamiltonian is given by,

$$H = \frac{p^2}{2} + V(x) + k \cos(x+1) \sum_{n=-\infty}^{+\infty} \delta(t - nT) \quad (70)$$

where  $V(x)$  is an infinite well potential of length  $\pi$ ,  $T$  is the period of the kick and  $k$  the strength. Concerning the classical motion, the KAM theorem is not applicable because the potential is not smooth. Indeed, it was found [54] that the classical phase space resembles a stochastic web with a self similar structure. This is in contrast with the standard kicked rotor where the classical phase space is a mixture of chaotic and integrable parts separated by KAM tori.

The quantum mechanical properties of the model are described by the evolution operator  $\hat{U} = e^{-i\hat{p}^2 T/4} e^{ik \cos(x+1)} e^{-i\hat{p}^2 T/4}$  over a period  $T$  of the kick. The quasi energies

associated to this operator were obtained in [54] by diagonalizing  $\hat{U}$  in a basis of 1024 eigenstates of the free Hamiltonian  $\langle q|n\rangle = \sqrt{\frac{2}{\pi}} \sin(nq)$ .

Unlike the kicked rotor where the matrix evolution has an exponential decay in a basis of plane waves, it can be shown that the matrix elements of  $\hat{U}$  are well described by a random banded matrix (RBM) with power-like decay,  $|\langle m|\hat{U}|m+n\rangle| \propto \frac{b^2}{n^2}$  for  $b \ll n$  and constant for  $b \gg n$  where  $b \sim K = kT$  is the size of the band.

In agreement with the results of [3], the nearest neighbor distribution reported in [54] smoothly interpolates between Poisson and *GOE* as  $K$  is increased. Recently, an experimental realization of this model was studied in [55].

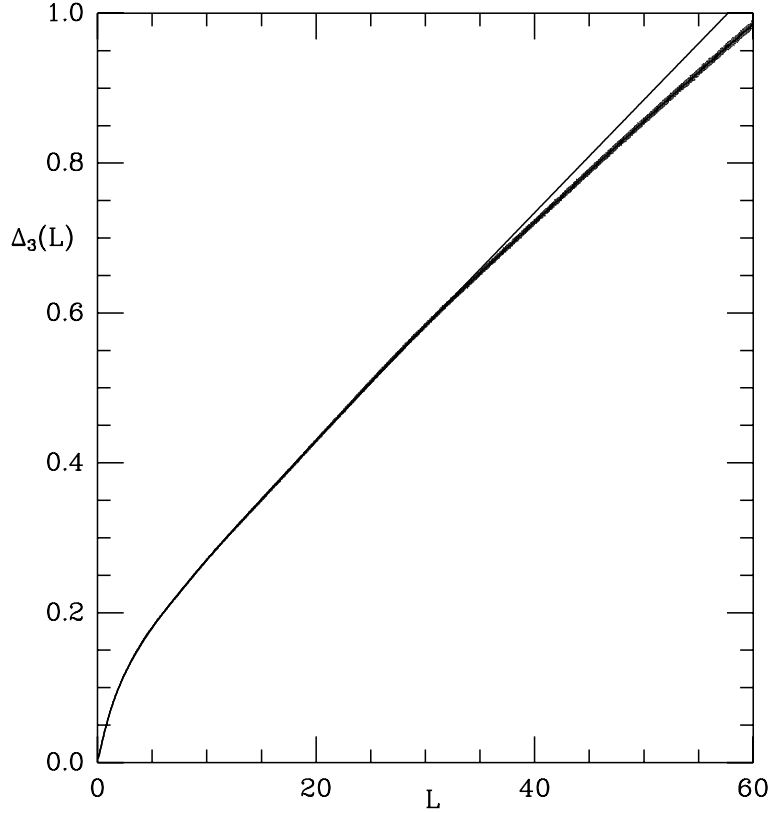


Figure 4: The spectral rigidity  $\Delta_3(L)$  obtained in [54] for the energy levels of a kicked particle in an infinite potential well at  $K = kT = 50$  is accurately described by the critical kernel (30) at  $h = 0.2122$  (upper curve). The error in the numerical results is indicated by the thickness of the curve.

### Analysis of results

In Figures 3 and 4, we show the number variance and the  $\Delta_3$ -statistic of the sequence of 1024 eigenvalues obtained in [54] for  $kT = 50$ . The upper curve is the analytical result

derived from the two-point correlation function (23) with kernel (30). In both cases the numerical result is plotted with its error (see below).

For the  $\Delta_3$ -statistic, the  $\chi^2$  is minimized on the interval  $[0, 30]$  for  $h = 0.2122$  with a value<sup>3</sup> of  $\chi^2 = 0.32$ . The errors in the definition of  $\chi^2$  have been calculated by splitting the 1024 eigenvalues into eight ensembles of 128 eigenvalues and evaluating the number variance for each ensemble separately (denoted by  $\Sigma_i^2(L)$ ,  $i = 1, \dots, 8$ ). The error in the number variance is thus given by

$$\sigma(L) = \frac{1}{\sqrt{8}} \left[ \frac{1}{8} \sum_{i=1}^8 (\Sigma_i^2(L) - \Sigma_{\text{mean}}^2(L))^2 \right]^{1/2} \quad \text{where} \quad \Sigma_{\text{mean}}(L) = \frac{1}{8} \sum_{i=1}^8 \Sigma_i^2(L). \quad (71)$$

The error in the  $\Delta_3$ -statistic is obtained from this error by means of a Monte-Carlo simulation using the relation between  $\Delta_3(L)$  and  $\Sigma^2(L)$  [56]. As observed in Figure 4, the error in  $\Delta_3(L)$  is much smaller than the error in  $\Sigma^2(L)$ .

Next we ask the question whether the asymptotic behavior of the spectral rigidity is linear without a logarithmic correction, just as in the analytical case. If a logarithmic correction is absent, one can almost discard the possibility of a mixed classical phase space as the reason of the observed deviation from the GOE. In order to prove the absence of such term we fit the numerical curve  $\Delta_3(L)_{\text{num}}$  to  $\Delta_3(L)_{\text{fit}} = a + bL + c \log L$ . For instance, on a interval  $[15, 25]$  a best fit is obtained for  $a = 0.110 \pm 0.004$ ,  $b = 0.0156 \pm 0.0001$ ,  $c = 0.002 \pm 0.002$  with a value of  $\chi^2 = 0.015$ . We find that the value of the coefficient  $c$  is compatible with zero. This suggests that the classical phase space is not a mixture of chaotic and integrable regions.

In the case of the number variance, because of the size of the error, no conclusive evidence on the absence of the logarithmic term can be obtained by such fit. At large distances the number variance seems to deviate from a linear behavior by a quadratic term. Although such terms are typically caused by finite size effects, we do not have a clear understanding of its origin. Since the  $\Delta_3$ -statistic projects out a quadratic dependence of the number variance the linear behavior persists to much larger distances in this case (see Figure 4).

Finally, let us confront the conjecture (23) with our present numerical results. As we mentioned previously, it may look reasonable to replace the kernel (30) by (56). It can be shown that the spectral rigidity obtained from the kernel (56) is almost indistinguishable from the one obtained from (30) and therefore the agreement with the numerical result is expected to be equally good with one fitting parameter at our disposal. However, a more careful analysis shows that the kernel (56) leads to a value of  $\chi^2$  much higher than the one obtained from (30). For the interval  $[0, 30]$ , using the kernels (30) and (56), a best fit

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<sup>3</sup>Here and below we find values of  $\chi^2$  that are significantly less than one. This is possible because the values of  $\Delta_3(L)$  for different values of  $L$  are correlated. Therefore, our values of  $\chi^2$  have to be used with care and cannot be interpreted in terms of a  $\chi^2$  distribution.

is obtained for  $h = 0.2122$  with a value of  $\chi_{\text{Fermi}}^2 = 0.32$  and for  $h' = 0.325$  with a value of  $\chi_{\text{Sqrt}}^2 = 2.27$ , respectively.

The above findings suggest that a self-similar classical phase space dominated by cantori has a strong impact on the quantum spectral correlations. Critical statistics appears as the leading candidate to describe such correlations and, consequently, enlarge the range of applicability of random matrix ensembles.

Finally, we list other quantum systems between integrable and chaotic whose quantum spectral correlations show a similarity with critical statistics: quantum billiards with a point scatterer [57, 58], the Kepler billiard [17, 59], semiconductor billiards [61, 62], the stadium billiard inside certain range of parameters [63]. For applications concerning pseudo integrable billiards we refer to [17].

## 6 Conclusions

In this article we have introduced a one parameter ensemble of symmetric random matrices. This ensemble interpolates between the Gaussian Orthogonal Ensemble and the Poisson ensemble and is capable of describing critical statistics.

We have shown that, in an eigenvalue basis, the joint eigenvalue distribution of our model coincides with the diagonal density matrix of the C-S model at finite temperature where the additional parameter of the matrix model plays the role of temperature. Remarkably, this equivalence can be extended to all random matrix ensembles associated with the large families of symmetric spaces according to the Cartan classification thus providing a novel link between strongly interacting quantum systems and random matrix theory.

We have calculated the spectral correlation functions based on a recent conjecture by Kravtsov and Tselik for the correlation functions of the C-S model in the low temperature limit. We have tested the validity of this conjecture by two independent methods: one based on the effective conformal symmetry of the C-S model in the low temperature limit and the other based on the thermodynamical properties of a gas of particles governed by the C-S Hamiltonian. We have found that both the long distance low temperature behavior of the two-point correlation function obtained from the conformal calculation and the susceptibility of the C-S model agree with the conjecture made by Kravtsov and Tselik.

Based on the Kravtsov-Tselik conjecture we find that, although level repulsion is still present, the number variance is asymptotically linear with a slope less than one and no subleading logarithmic term present. This indicates that our random matrix model describes critical statistics.

Finally, we have argued that critical statistics is relevant to describe spectral corre-



lations of chaotic quantum systems for which the Poincare section of the classical counterpart is globally self-similar/fractal. Two examples with such classical phase space, a kicked particle in a potential well and the anisotropic Kepler problem, have been discussed in detail. In both cases, long range spectral correlators such as the number variance and the  $\Delta_3$ -statistic are accurately described by our analytical results based on the Kravtsov-Tsvelik conjecture. Indeed, for the kicked particle, we have shown that the spectral rigidity is asymptotically linear with no subleading logarithmic term present. This may be an indication that the wavefunctions of this model show multifractal properties.

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